$\mathcal{PT}-$symmetric regularizations in supersymmetric quantum mechanics

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Abstract

Within the supersymmetric quantum mechanics the necessary regularization of the poles of the superpotentials on the real line of coordinates x may be most easily mediated by a small constant shift of this axis into complex plane. Detailed attention is paid here to the resulting recipe which works, in effect, with non-Hermitian (a. k. a. $\mathcal{PT}-$symmetric or pseudo-Hermitian) Hamiltonians. Besides an exhaustive discussion of the role of the complex spike in harmonic oscillator, we mention also some applications concerning the regularized versions of the Smorodinsky-Winternitz and Calogero models and of the relativistic Klein-Gordon equation.

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1 Introduction: Symmetries

The concept of symmetry proves extremely productive in various branches of physics. In particular, its applications in quantum mechanics simplify the analysis of the properties of a symmetric system characterized by the commutativity of its Hamiltonian H_{SY} with some operators $S = S_{SY}$ which are tractable as elements of a Lie algebra of symmetries $\mathcal{G}_{(SY)}$. Such an additional information about the Schrödinger bound-state problem simplifies its solution. For example, during the construction of bound states in a $D-$dimensional central potential, the use of its symmetry $\mathcal{G}_{(SY)} = so(D)$ enables us to reduce the *partial* differential Schrödinger equation to its *exactly solvable* angular part $S_{(SY)}\,|n\rangle = \ell(\ell+1)\,|n\rangle$, accompanied by the *ordinary* radial differential equation.

A schematic illustrations of such a method is provided by the $D = 1$ Hamiltonian

$$H_{SY} = -\frac{d^2}{dx^2} + V_{SY}(x), \qquad x \in (-\infty, \infty) \tag{1}$$

where the real and confining potential is chosen as spatially symmetric, $V_{SY}(x) = V_{SY}(-x)$. This means that the Hamiltonian itself commutes with the operator of the parity $S_{SY} \equiv \mathcal{P}$. Each bound state $|n\rangle$ must be an eigenstate of both the Hamiltonian H_{SY} *and* the operator(s) S_{SY} so that we may select its parity in advance and, then, evaluate the wavefunction $|n\rangle$ on the mere half-axis of $x \in (0, \infty)$.

Beyond the similar Lie-algebraic applications, a natural generalization of the concept of symmetry is encountered in the so called supersymmetric quantum mechanics (SUSY QM, cf. the review paper [1]) and in the so called $\mathcal{PT}-$symmetric quantum mechanics (PTSY QM, cf. its recent formulations in Refs. [2, 3]). In these two cases, the linear Lie algebra of symmetries $\mathcal{G}_{(SY)}$ is being replaced by a graded Lie algebra $\mathcal{G}_{(SUSY)}$ and by its antilinear (i.e., strictly speaking, non-linear) analogue $\mathcal{G}_{(PTSY)}$, respectively. Here we intend to pay attention to the various combinations of these two possibilities.

In the former context, transition to SUSY QM is based on a deeper mathematical interpretation of the creation and annihilation operators. On a less abstract level, SUSY QM works with the doublets of the [say, "left" (or $_L-$subscripted) and "right" (or $_R-$subscripted)] Hamiltonians arranged in a single super-Hamiltonian

$$H_{SUSY} = \left[\begin{array}{cc} H_{(L)} & 0 \\ 0 & H_{(R)} \end{array} \right], \qquad H_{(L,R)} = -\frac{d^2}{dx^2} + V_{(L,R)}(x) \tag{2}$$

which commutes with the two matrix operators which are called super-charges,

$$S_{SUSY,A} = \left[\begin{array}{cc} 0 & 0 \\ A & 0 \end{array} \right] \equiv \mathcal{Q}, \qquad S_{SUSY,B} = \left[\begin{array}{cc} 0 & B \\ 0 & 0 \end{array} \right] \equiv \tilde{\mathcal{Q}}\,. \tag{3}$$

In 1981, E. Witten [4] noticed the relevance of such a model for clarification of the absence of any SUSY partners of the observable elementary particles. He tried to attribute their "invisibility in experiments" to the spontaneous breakdown of SUSY.

Paradoxically, during the last cca twenty years, the idea found the majority of its applications within non-relativistic quantum mechanics [5, 6].

The history of the SUSY-inspired search for a symmetry between bosons and fermions is not too dissimilar to the recent development of PTSY QM which also finds some of its key physical motivations within the relativistic quantum field theory [7]. Many open questions concern the mathematics of PTSY and, in particular, a deeper understanding of its spontaneous breakdown [8]. Simplified PTSY QM models are studied, based on the presence of the two real potentials in a single, *non-Hermitian* Hamiltonian

$$H_{PTSY} = -\frac{d^2}{dx^2} + V_S(x) + i\,V_A(x), \qquad x \in (-\infty, \infty)\,. \tag{4}$$

The two components $V_S(x) = V_S(-x)$ and $V_A(x) = -V_A(-x)$ of the complexified force are, by assumption, spatially symmetric and antisymmetric functions, respectively. The Hamiltonian commutes, this time, with the product $S_{PTSY} \equiv \mathcal{P} \cdot \mathcal{T}$ of the parity $\mathcal{P}$ with an auxiliary antilinear operator $\mathcal{T}$ which mimics the time reversal [2].

In a broader context, operator $\mathcal{P}$ may be generalized to any Hermitian and invertible "metric" $\mathcal{P} = \mathcal{P}^\dagger$ (sometimes assumed to be positive definite [9]) while its partner $\mathcal{T}$ mediates Hermitian conjugation [10, 11]. This means that we may eliminate the explicit use of $\mathcal{T}$ and treat the PTSY-related commutativity

$$S_{PTSY} H_{PTSY} - H_{PTSY} S_{PTSY} = 0 \tag{5}$$

as a mere pseudo-Hermiticity requirement [12]

$$H^\dagger_{PTSY} = \mathcal{P}\, H_{PTSY}\, \mathcal{P}^{-1}\,. \tag{6}$$

The best known phenomenological, *experimentally* relevant Hamiltonians with the "weakened Hermiticity" property (6) appear in the Feshbach-Villars version of the relativistic Klein-Gordon equation (cf. section 5.2 below). More than ten years ago, pseudo-Hermitian Hamiltonians also became popular in many phenomenological descriptions of the bosonic degrees of freedom within atomic nuclei [9].

We may summarize that the common Lie-algebraic S_{SY} finds its close parallels within SUSY QM and PTSY QM. In the present paper, we intend to develop, in due detail, several ideas which emerged in our papers [13]-[15] where we outlined some merits of a synchronized work within a *combined* $\mathcal{PT}-$symmetric *and* superymmetric quantum mechanics (PTSUSY QM).

2 Conventional regularizations in SUSY QM

The key SUSY assumption of commutativity may be read, alternatively, as a factorization postulate for $\mathcal{H}$,

$$\left[\begin{array}{cc} H_{(L)} & 0 \\ 0 & H_{(R)} \end{array} \right] = \left[\begin{array}{cc} BA & 0 \\ 0 & AB \end{array} \right]. \tag{7}$$

This postulate implies the further relations

$$\{\mathcal{Q},\tilde{\mathcal{Q}}\}=\mathcal{H},\qquad \{\mathcal{Q},\mathcal{Q}\}=\{\tilde{\mathcal{Q}},\tilde{\mathcal{Q}}\}=0,\qquad [\mathcal{H},\mathcal{Q}]=[\mathcal{H},\tilde{\mathcal{Q}}]=0$$

which, in their turn, enable us to treat the supercharges S_{SUSY} as generators of the graded Lie algebra sl(1/1).

Example (1) with the $D-$dimensional harmonic-oscillator $V_{(SY)}^{(HO)}(|\vec{r}|)=|\vec{r}|^2$ offers one of the most transparent illustrations of the essence of many of the characteristic ideas of SUSY QM. Thus, at $D=1$ one inserts the two differential operators $A=\partial_x+x$ and $B=-\partial_x+x$ of the first order in the explicit definition (3) of the two supercharge operators. The resulting explicit SUSY Hamiltonian (2) connects the two *regular* harmonic oscillators $H_{(L,R)}^{(HO)}=-d^2/dx^2+x^2\mp 1$. The wave functions pertaining to these oscillators are analytic and may serve, in parallel, as an illustration of a transition to the complexified coordinates x in PTSY QM of Ref. [2].

After a transition to $D>1$, equation (1) remains almost the same, complemented merely by a strongly singular kinetic term in $V_{SY}(x)\to V_{SY}(x)+V_{kinetic}(x)$ where $V_{kinetic}(x)=\ell(\ell+1)/x^2$ with $\ell=\ell(n)=(D=3)/2+n$ and $n=0,1,\ldots$ [16]. Marginally, it is rather amusing to notice that precisely the difference between the absence and presence of the strongly singular $V_{kinetic}(x)$ plays a key role in SUSY QM. Indeed, one of the characteristic features of the formalism of SUSY QM is the requirement of the absence of the real poles in its partner potentials $V_{(L,R)}$ (see, e.g., the review paper [1] for more details). *Vice versa*, a puzzling difficulty is immediately encountered when we try to study a SUSY model with such a singularity. During the detailed analysis of the central harmonic oscillator (HO) in $D>1$ dimensions, Jevicki and Rodrigues (JR, [17]) revealed that in such a singular case, the textbook recipes lead to a *singular* form of the HO superpotential,

$$W(x)=W^{(\gamma)}(x)=x-\frac{\gamma+1/2}{x} \tag{8}$$

where γ may vary with D etc [14]. Then, at any real strength parameter γ, the naive SUSY algebra is well known to generate the formal partnership between the two harmonic oscillator Hamiltonian operators of the form $H^{(\kappa)}=-d^2/dx^2+x^2+(\kappa^2-1/4)/x^2$ where $\kappa=\kappa_{L,R}=\ell_{L,R}+1/2$ [1]. Of course, in order to satisfy the two factorization requirements in eq. (7) we must define

$$H_{(L)}=H^{(\alpha)}-2\gamma-2,\qquad H_{(R)}=H^{(\beta)}-2\gamma \tag{9}$$

and put $\alpha=|\gamma|=\ell_{(L)}+1/2$ and $\beta=|\gamma+1|=\ell_{(R)}+1/2$. This means that people usually restrict their attention to the regular or "linear" HO Hamiltonians $H^{(\pm 1/2)}$ defined on the whole axis of $x\in(-\infty,\infty)$ and possessing the well known energy spectrum

$$E_n^{(LHO)}=2n+1,\qquad n=0,1,\ldots\,. \tag{10}$$

In such a regular case (i.e., at the exceptional $\gamma=-1/2$) the partner SUSY spectra $[E_{(L,R)}^{(\gamma)}]_n$ reflect the shifts in eq. (9) and exhibit the expected SUSY-type isospectrality,

$$E_{(L)}^{(-1/2)}=\{0,2,4,6,8,\ldots\},\qquad E_{(R)}^{(-1/2)}=\{2,4,6,8,\ldots\}\,. \tag{11}$$

More rarely, one may arrive at the similar conclusions using the *more general* or "singular" HO Hamiltonian operators $H^{(\kappa)}$ which *must be restricted* to the smaller domain of $x \in (0,\infty)$ [or, if necessary, $x \in (-\infty,0)$] and possess, therefore, the *different* energy spectrum $E_n^{(SHO)} = 4n + 2\ell + 3$.

On this background it is easy to characterize the essence of the puzzle described by Jevicki and Rodrigues [17] as an attempted study of the relationship between the *same* operators defined on the *different* domains (see Ref. [14] for more details). For illustration of the essence of this puzzle, it is most popular to pick up the $\gamma \neq -1/2$ superpotential (8), say, with $\gamma = 1/2$. Then, the constant shifts in eq. (9) give us immediately the very different "partner" spectra

$$E_{(L)}^{(1/2)} = \{-2, 0, 2, 4, 6, 8, \ldots\}, \qquad E_{(R)}^{(1/2)} = \{4, 8, 12, 16, \ldots\}\,. \tag{12}$$

The explanation of the sudden loss of the desired SUSY-type isospectrality lies in the *one-sided* emergence of the centrifugal-like singularity $2/x^2$ in the right SUSY partner $H_{(R)}$ only. At the same time, the left operator $H_{(L)}$ remains regular on the whole real line. The complete set of the left normalizable wave functions is not constrained by the boundary condition applied to the right wavefunctions in the origin, $\psi_{(R)}(0) = 0$. This observation might inspire a naive resolution of the JR paradox, based on an addition of the artificial and arbitrary left boundary condition $\psi_{(L)}(0) = 0$. This would reduce the left spectrum to the set $E_{(L)} = \{0, 4, 8, \ldots\}$ which obeys the required SUSY isospectrality rule.

An alternative, less naive though much more cumbersome regularization recipe has been offered by Das and Pernice [18] who re-defined slightly $H_{(L)}$ and $H_{(R)}$ via an additional delta function in the origin. In effect, their construction proves equivalent [15] to the use of the Dirichlet boundary condition $\psi_{(R)}(0) = 0$ for the right Hamiltonian *in a rather unusual combination* with the Neumann left boundary condition, $\partial_x \psi_{(L)}(0) = 0$. They obtained

$$E_{(L)}^{(1/2)} = \{0, 2, 4, 6, 8, \ldots\}, \qquad E_{(R)}^{(1/2)} = \{2, 4, 6, 8, \ldots\}\,. \tag{13}$$

In our present paper we intend to emphasize that a much easier and more natural approach to the resolution of the JR regularization puzzle may be based on a downward complex shift of the real axis of x. Besides giving an outline of its essence (i.e., of an analytic PTSY extension of quantum mechanics), we shall also mention some of the possible consequences of this approach for a few other important SUSY-related models with singularities.

3 Regularized constructions within PTSUSY QM

3.1 An exactly solvable model

Harmonic-oscillator Hamiltonians

$$H^{(\alpha)} = -\frac{d^2}{dr^2} + \frac{\alpha^2 - 1/4}{r^2} + r^2, \qquad \alpha > 0$$

with the Buslaev's and Grecchi's [16] PTSY regularization of $r = x - i\,\varepsilon$ (with real x, cf. also Ref. [19] in this context) have thoroughly been studied in Ref. [20]. Their normalizable wave functions

$$\langle r|N^{(\varrho)}\rangle = \frac{N!}{\Gamma(N+\varrho+1)} \cdot r^{\varrho+1/2} \exp(-r^2/2) \cdot L_N^{(\varrho)}(r^2) \tag{14}$$

and energies

$$E = E_N^{(\varrho)} = 4N + 2\varrho + 2, \qquad \varrho = -Q \cdot \alpha$$

were described there as manifestly dependent on the quantum number $Q = \pm 1$ of the so called quasi-parity. In the context of fields, this quantity proves intimately related to the more recent concept of the so called charge-conjugation symmetry $\mathcal{C}$ as introduced, e.g., in Ref. [10].

We already mentioned that the regularized superpotentials $W^{(\gamma)}$ enter the SUSY operators $A^{(\gamma)} = \partial_q + W^{(\gamma)}$ and $B^{(\gamma)} = -\partial_q + W^{(\gamma)}$ in a way which defines the "left" and "right" components of the super-Hamiltonian $\mathcal{H}$,

$$H_{(L)} = B \cdot A = \hat{p}^2 + W^2 - W', \qquad H_{(R)} = A \cdot B = \hat{p}^2 + W^2 + W'.$$

For our present choice of $W = W^{(\gamma)}$ these SUSY partners are $\gamma-$dependent and, generically (i.e., at any complex γ), both of them are proportional to the *different* harmonic oscillators,

$$H_{(L)}^{(\gamma)} = H^{(\alpha)} - 2\gamma - 2, \qquad H_{(R)}^{(\gamma)} = H^{(\beta)} - 2\gamma, \qquad \alpha^2 = \gamma^2, \qquad \beta^2 = (\gamma+1)^2 \tag{15}$$

with, conventionally, Re $\alpha > 0$ and Re $\beta > 0$.

3.2 SUSY under unbroken $\mathcal{PT}$ symmetry

At all the real $\gamma \in (-\infty, \infty)$ we select $\alpha > 0$ and $\beta > 0$. This means that all the energies remain real and that one may distinguish between the following three regimes,

$$\left\{ \begin{array}{ll} \text{positive } \gamma = \alpha > 0, & \text{dominant } \beta = \alpha + 1, \\ \text{small negative } \gamma = -\alpha > -1, & \text{comparability, } \alpha + \beta = 1, \\ \text{large negative } \gamma = -\alpha < -1, & \text{dominant } \alpha = \beta + 1. \end{array} \right. \tag{16}$$

For a given "left" Hamiltonian $H_{(L)} \sim H^{(\alpha)}$ we have a choice between $\gamma = \pm\alpha$ giving the two alternative "right" partners $H^{(\beta_{1,2})}$ such that $\beta_{1,2} = 1 \pm \alpha$ for small $\alpha < 1/2$ while $\beta_1 = |\alpha - 1| < \alpha < \beta_2 = \alpha + 1$ for $\alpha > 1/2$. Each pair of the SUSY partners $H_{(L,R)}^{(\gamma)}$ generates the quasi-even and quasi-odd energies and these energies form an ordered quadruplet at any main quantum number n,

$$\left\{ \begin{array}{lllll} E_{(L)}^{(-\alpha)} & = E_{(R)}^{(-\beta)} & < E_{(L)}^{(+\alpha)} & < E_{(R)}^{(+\beta)} & \text{for positive } \gamma, \\ E_{(L)}^{(-\alpha)} & < E_{(R)}^{(-\beta)} & = E_{(L)}^{(+\alpha)} & < E_{(R)}^{(+\beta)} & \text{for negative } \gamma > -1, \\ E_{(L)}^{(-\alpha)} & < E_{(R)}^{(-\beta)} & < E_{(L)}^{(+\alpha)} & = E_{(R)}^{(+\beta)} & \text{for negative } \gamma < -1 \end{array} \right. \tag{17}$$

(mind and mend the misprints in Table 1 of Ref. [14] and consult also Figure 1 *ibidem*). The equal-n degeneracies may be emphasized by the square brackets "[" and "]",

$$\left\{ \begin{array}{ll} [4n-4\alpha] < 4n < 4n+4 & \text{for positive } \gamma, \\ 4n < [4n+4\alpha] < 4n+4 & \text{for negative } \gamma > -1, \\ 4n < 4n+4 < [4n+4\alpha] & \text{for negative } \gamma < -1. \end{array} \right. \tag{18}$$

By their (linear) $\alpha-$dependence, they are distinguished from the usual non-equal-n degeneracies with the values of energies which do not vary with this parameter.

3.3 Spontaneously broken symmetry regime

The $\mathcal{PT}$ symmetry of the "left" Hamiltonians $H_{(L)} \sim H^{(\alpha)}$ becomes spontaneously broken at the purely imaginary α [20]. This results from the purely imaginary choice of γ in the superpotential $W^{(\gamma)}$. We have to distinguish between the two separate half-lines,

$$\left\{ \begin{array}{lll} \delta > 0 \text{ in } \gamma = i\,\delta, & \alpha = i\,\delta, & \beta = 1+\alpha, \\ \eta > 0 \text{ in } \gamma = -i\,\eta, & \alpha = i\,\eta, & \beta = 1-\alpha. \end{array} \right. \tag{19}$$

On both of them the energies pertaining to the unshifted and $\mathcal{PT}$ symmetric Hamiltonian $H^{(\alpha)}$ occur in the complex conjugate pairs $E_n^{(\pm\alpha)}$ [21]. This symmetry is broken by the shift $H^{(\alpha)} \to H^{(\gamma)}_{(L)}$ enforced by the SUSY factorization in eq. (15). Hence, the supersymmetrization still generates the partially real spectra even at the purely imaginary $\alpha = \sqrt{\gamma^2}$,

$$\left\{ \begin{array}{llll} E^{(+\alpha)}_{(L)} = 4n, & E^{(-\alpha)}_{(L)} = E^{(-\beta)}_{(R)} = 4n - 4\alpha, & E^{(+\beta)}_{(R)} = 4n+4 & \text{for } \gamma = i\,\delta, \\ E^{(-\alpha)}_{(L)} = 4n, & E^{(+\alpha)}_{(L)} = E^{(-\beta)}_{(R)} = 4n + 4\alpha, & E^{(+\beta)}_{(R)} = 4n+4 & \text{for } \gamma = -i\,\eta. \end{array} \right.$$

We see that the imaginary components either vanish or are equal to 4γ. In the latter case the complex "left" and "right" energies coincide at a fixed n. The change of the sign of γ causes their complex conjugation. Simultaneously, the real parts of the energies are $\gamma-$independent and equidistant, and the characteristic degeneracy pattern $E_{(L)}(n) = E_{(R)}(n-1)$ holds for all the *real* levels in the spectrum exempting, as usual, the non-degenerate $n = 0$ state.

3.4 Completely broken symmetry regime

We have seen that the spontaneous breakdown of the $\mathcal{PT}$ symmetry of our initial or "left" Hamiltonian $H_{(L)}$ leads to the mere *partial* complexification of the related energies $E_{(L)}$. One may reformulate this observation by saying that the SUSY construction keeps some "right" energies $E_{(R)}$ real even though the $\mathcal{PT}$ symmetry of the "right" Hamiltonian $H_{(R)}$ itself is violated in the *manifest* manner.

The latter empirical rule may be easily generalized by induction. For this purpose let us choose a positive integer N and start from the superpotential $W^{(\gamma)}$ with a complex parameter $\gamma = N + i\,q\,\delta$ with $\delta > 0$ and $q = \pm 1$. In the first step we get

the SUSY rules $\alpha^2 = \gamma^2$ and $\beta^2 = (\gamma+1)^2$ and make our α and β unique by the constraints Re $\alpha > 0$, Im $\alpha = \delta > 0$ and Re $\beta > 0$. This gives the two options,

$$\beta_1 = N - 1 + i\,\delta < \alpha = N + i\,\delta < \beta_2 = N + 1 + i\,\delta$$

i.e., either a return to the previous N or the induction step towards the next one. Now we may summarize that at all these N the SUSY rules lead, up to the ambiguity in signs, to the same conclusion as above, giving

$$E_{(L)} = \left\{ \begin{array}{l} 4n - 4\gamma \\ 4n \end{array} \right. , \qquad E_{(R)} = \left\{ \begin{array}{l} 4n - 4\gamma \\ 4n + 4 \end{array} \right. . \tag{20}$$

In this way, the picture is closed and complete: The class of the spiked harmonic oscillator SUSY partners possessing the same coinciding and partially real spectra (20) may be all derived from the superpotential $W^{(\gamma)}$ defined on the lattice of $\gamma = N \pm i\,\delta$ with integers $N = 0, 1, \ldots$ and positive $\delta > 0$. The $N = 0$ observations of subsection 3.3 remain valid at all N. Also the $\mathcal{PT}$ symmetric limit $\delta \to 0$ moves us to the exceptional points $\alpha =$ integer where one encounters the Jordan-block structures and unavoided crossings of the energy levels [22, 20, 23].

4 An alternative supersymmetrization

The most striking feature of the above SUSY scheme lies in the puzzling jumps between the different Hamiltonians $H^{(\alpha)}$ and $H^{(\beta)}$. The action of the linear differential operators $A^{(\gamma)} = \partial_q + W^{(\gamma)}$ and $B^{(\gamma)} = -\partial_q + W^{(\gamma)}$ on the states $|n^{(\gamma)}\rangle$ may change both their quantum number n *and* the parameter γ.

A transition from γ to quasi-parity Q and/or to the two "physical" spike-strength parameters α and β clarifies that the "natural" connection between the SUSY and annihilation and creation operators remains internally consistent in the exceptional $\alpha \to 1/2$ limit only. Fortunately, in all the other cases, a return $\beta \to \alpha$ to the original system may still be mediated by elementary operators. The determination of their action on the Laguerre polynomials appearing in the explicit form of our present wave functions is easy and the rule we need is given by the quadratic operators

$$\mathbf{B}(\alpha) = B^{(-\gamma)} \cdot B^{(\gamma-1)} \;\equiv\; B^{(\gamma)} \cdot B^{(-\gamma-1)}, \qquad \mathbf{A}(\alpha) = A^{(-\gamma-1)} \cdot A^{(\gamma)} \;\equiv\; A^{(\gamma-1)} \cdot A^{(-\gamma)}$$

of Ref. [14] (cf. also [24] and [25]) which satisfy the relations

$$\mathbf{B}(\alpha)\, \left|N^{(\gamma)}\right\rangle = c(N,\gamma)\, \left|(N+1)^{(\gamma)}\right\rangle, \qquad \mathbf{A}(\alpha)\, \left|(N+1)^{(\gamma)}\right\rangle = c(N,\gamma)\, \left|N^{(\gamma)}\right\rangle$$

where $\gamma = \pm\alpha$ may now be complex while the formula $c(N,\gamma) = -4\sqrt{(N+1)(N+\gamma+1)}$ of Ref. [14] remains unchanged.

The latter equations may be interpreted as the respective creation and annihilation of a state at a quasi-parity $Q = -$sign (Re γ). One may, if necessary, exempt here the anomalous integers $\alpha = \alpha_c = 0, 1, 2, \ldots$ at which one of the coefficients

$c(N,\gamma)$ vanishes and where the two different energy levels cross or, in other words, where the Bender-Wu singularities [26] become real.

A possible deeper meaning of the peculiar second-order differential operators $\mathbf{B}(\alpha)$ and $\mathbf{A}(\alpha)$ is to be sought now by the straightforward calculation which reveals that

$$\mathbf{A}(\alpha)\,\mathbf{B}(\alpha)\;-\;\mathbf{B}(\alpha)\,\mathbf{A}(\alpha)\;\equiv\;8\,H^{(\alpha)}.$$

This means that in spite of their unusual form, our innovated creation and annihilation operators exhibit an unexpected simplicity. A "hidden" algebraic meaning of the above rule may be significantly clarified by the explicit evaluation the two further commutators,

$$\mathbf{A}(\alpha)\,H^{(\alpha)}\;-\;H^{(\alpha)}\,\mathbf{A}(\alpha)\;\equiv\;4\,\mathbf{A}(\alpha),\qquad H^{(\alpha)}\,\mathbf{B}(\alpha)\;-\;\mathbf{B}(\alpha)\,H^{(\alpha)}\;\equiv\;4\,\mathbf{B}(\alpha).$$

This implies that the three differential operators $\mathbf{A}(\alpha)/\sqrt{32}$, $\mathbf{B}(\alpha)/\sqrt{32}$ and $H^{(\alpha)}/4$ of the second order are generators of the Lie algebra $sl(2,R)$.

By the existence of the symmetries just revealed, we feel encouraged to define the new set of operators

$$\mathbf{G}=\left[\begin{array}{cc}\mathbf{G}_{(L)} & 0\\ 0 & \mathbf{G}_{(R)}\end{array}\right],\qquad \mathbf{Q}=\left[\begin{array}{cc}0 & 0\\ \mathbf{A}(\alpha) & 0\end{array}\right],\qquad \tilde{\mathbf{Q}}=\left[\begin{array}{cc}0 & \mathbf{B}(\alpha)\\ 0 & 0\end{array}\right]\tag{21}$$

which may be perceived as a modified or alternative SUSY recipe. Once we put

$$\mathbf{G}_{(L)}=\mathbf{B}(\alpha)\,\mathbf{A}(\alpha)\,,\qquad \mathbf{G}_{(R)}=\mathbf{A}(\alpha)\,\mathbf{B}(\alpha)\tag{22}$$

the new operators (21) generate a representation of the SUSY algebra sl(1/1),

$$\{\mathbf{Q},\tilde{\mathbf{Q}}\}=\mathbf{G},\qquad \{\mathbf{Q},\mathbf{Q}\}=\{\tilde{\mathbf{Q}},\tilde{\mathbf{Q}}\}=0,\qquad [\mathbf{G},\mathbf{Q}]=[\mathbf{G},\tilde{\mathbf{Q}}]=0.$$

The elements of the innovated super-Hamiltonian-like two-by-two matrix $\mathbf{G}$ are differential operators of the fourth order. They are defined, in the most compact form, by the products (22) of the two boldface and mutually non-adjoint factors

$$\mathbf{B}(\alpha)=\frac{d^2}{d\,x^2}-\frac{\alpha^2-1/4}{(x-i\,\varepsilon)^2}+(x-i\,\varepsilon)^2-2\,(x-i\,\varepsilon)\,\frac{d}{d\,x}-1,$$

$$\mathbf{A}(\alpha)=\frac{d^2}{d\,x^2}-\frac{\alpha^2-1/4}{(x-i\,\varepsilon)^2}+(x-i\,\varepsilon)^2+2\,(x-i\,\varepsilon)\,\frac{d}{d\,x}+1.$$

The action of both the components of $\mathbf{G}$ on our basis states is transparent,

$$\mathbf{G}_{(L)}\,\left|N^{(\gamma)}\right\rangle=\Omega_N^{(\gamma)}\,\left|N^{(\gamma)}\right\rangle\,,\qquad \Omega_N^{(\gamma)}=16\,N\,(N+\gamma)$$

$$\mathbf{G}_{(R)}\,\left|N^{(\gamma)}\right\rangle=\Omega_{N+1}^{(\gamma)}\,\left|N^{(\gamma)}\right\rangle\,.$$

Tractable as the respective creation and annihilation operators, both $\mathbf{B}(\alpha)$ and $\mathbf{A}(\alpha)$ conserve the quasi-parity Q (i.e., the sign of the superscript γ) and enable us to split the Hilbert space into two separate "halves" which may be marked by $Q=\pm 1$.

5 Towards realistic applications

One of the key messages of our present paper is its emphasis on the productivity of a combination of the algebraic ideas of supersymmetry [illustrated by the factorization (7) of some elementary harmonic-oscillator-type differential operators] with the ideas of $\mathcal{PT}-$symmetry and analytic continuation (best exemplified by the singularity paradoxes and their comparatively easy analytic regularizations in complex plane). Several merits of such a combined approach may be emphasized. First of all, one becomes able to reveal an algebraic background of the analytic properties (e.g., an analyticity-based complex shift of a complex coordinate found its algebraic pseudo-Hermitian explanation in Refs. [19, 21]) and vice versa (for example, the algebraic "spectral-equivalence" effect of the same shift has found its analytic-continuation explanation in Refs. [2, 16] or [20]).

Of course, our natural ambition of extending the similar parallels beyond the domain of the most elementary illustrative examples is limited by the numerous technical obstacles. In the second half of our paper let us pay attention to their selected sample.

5.1 Angular Schrödinger equations in PTSUSY QM

In the introductory part of the review [1] (cf. the picture Nr. 2.2 there) it has been emphasized that the free motion within a square well generates the whole SUSY family of the exactly solvable models, the first nontrivial element of which is the single-well Pöschl-Teller problem on a finite interval,

$$H_{1W} = -\frac{d^2}{d\varphi^2} + V_{1W}(\varphi), \qquad V_{1W}(\varphi) = \frac{A(A+1)}{\sin^2 \varphi}, \qquad \varphi \in (0, \pi)\,. \tag{23}$$

Bagchi et al [27] have emphasized that the same observation remains valid when we consider the PTSY generalization of this model.

It has been noticed in our recent paper [28] that the merely slightly modified angular differential equation

$$H_{4W} = -\frac{d^2}{d\varphi^2} + V_{4W}(\varphi), \qquad V_{4W}(\varphi) = \frac{A(A+1)}{\sin^2 4\,\varphi}, \qquad \varphi \in (0, \pi/4) \tag{24}$$

is a core of the construction of the bound states in the non-central potentials

$$V^{(SI)}(X,Y) = X^2 + Y^2 + \frac{G}{X^2} + \frac{G}{Y^2} \tag{25}$$

of the Smorodinnsky-Winternitz "superintegrable" family in two dimensions [29]. In parallel, the famous Calogero's [30] three-body bound-state model represented by the separable partial differential equation

$$\left[-\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + X^2 + Y^2 + \right.$$

$$+\frac{g}{2\,X^2}+\frac{g}{2\,(X-\sqrt{3}Y)^2}+\frac{g}{2\,(X+\sqrt{3}Y)^2}\Bigg]\,\Phi(X,Y)=\mathcal{E}\,\Phi(X,Y) \tag{26}$$

has a very similar ordinary differential angular part,

$$H_{6W}=-\frac{d^2}{d\varphi^2}+V_{6W}(\varphi), \qquad V_{6W}(\varphi)=\frac{A(A+1)}{\sin^2 6\,\varphi}, \qquad \varphi\in(0,\pi/6)\,. \tag{27}$$

Unfortunately, the $\mathcal{PT}-$symmetrization of the latter two models does not lead to any simplifications similar to those which we witnessed, e.g., in section 3.1 above. At this point, it is our sad duty to say that an opposite observation is true. One of the main reasons is that in the Hermitian version of both the above-mentioned models (where the Pöschl-Teller force appears in the angular part of the Schrödinger equation), Dirichlet boundary conditions are imposed at both the ends of the interval of the angle φ. This reflects the presence of strongly repulsive barriers at both these exceptional points.

After the PTSY regularization of the coordinate we arrive at the different family of eigenvalue problems,

$$H_{MW}=-\frac{d^2}{d\varphi^2}+V_{MW}(\varphi), \qquad V_{MW}(\varphi)=\frac{A(A+1)}{\sin^2 M\,\varphi}, \tag{28}$$

$$M=1,2,\ldots, \qquad \varphi=\alpha-i\,\varepsilon, \qquad \alpha\in(0,\pi)\,.$$

Our choice of the complex shift $\varepsilon>0$ opens the possibility of the tunnelling through the barriers (cf. the particular $M=4$ and $M=6$ results in Refs. [31] and [32], respectively). This means that we must impose the different, periodic boundary conditions at the endpoints of our domain of the angular coordinate α which is M times bigger.

The phenomenon of the tunnelling between the neighboring (sometimes called "Weyl's") chambers is, in this context, a source of a new physics as well as of a significantly worsened mathematical insight in the structure of the PTSY solutions at any $M\geq 2$. This has been illustrated on a schematic semi-numerical double-well model in Ref. [28].

5.2 Relativistic PTSUSY quantum mechanics?

Once we moved to the non-Hermitian Hamiltonians, the scope of quantum mechanics (including also its specific regularization aspects) finds a fairly natural extension to the domains with relativistic kinematics.

In order to be more specific, let us pick up the well known example of the the Klein-Gordon equation in the units $\hbar=c=1$,

$$(i\,\partial_t)^2\,\Psi^{(KG)}(x,t)=H^{(KG)}\,\Psi^{(KG)}(x,t)\,. \tag{29}$$

This equation is not too different from its Schrödinger non-relativistic predecessor

$$i\,\partial_t\Psi^{(NR)}(x,t)=H^{(NR)}\,\Psi^{(NR)}(x,t)\,. \tag{30}$$

After we abbreviate $\varphi_1 = i\,\partial_t\,\Psi^{(KG)}(x,t)$ and $\varphi_2 = \Psi^{(KG)}(x,t)$, the relativistic evolution may be described by the equivalent Feshbach-Villars [33] first-order differential equation presented here in the form

$$i\,\partial_t \left(\begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right) = \left(\begin{array}{cc} 0 & H^{(KG)} \\ 1 & 0 \end{array} \right) \cdot \left(\begin{array}{c} \varphi_1 \\ \varphi_2 \end{array} \right) . \tag{31}$$

In contrast to the freedom of option between Hermiticity and pseudo-Hermiticity for the current non-matrix Hamiltonians,

$$\left[H^{(NR,KG)}\right]^\dagger = \eta\, H^{(NR,KG)}\,\eta^{-1}, \qquad \eta = \eta^\dagger\,, \tag{32}$$

the two-by-two Feshbach-Villars matrix operator and equation prove manifestly non-Hermitian,

$$\left[H^{(FV)}\right]^\dagger = \mathcal{P}\,H^{(FV)}\,\mathcal{P}^{-1}, \qquad H^{(FV)} = \left(\begin{array}{cc} 0 & H^{(KG)} \\ 1 & 0 \end{array} \right), \qquad \mathcal{P} = \left(\begin{array}{cc} 0 & \eta \\ \eta & 0 \end{array} \right) . \tag{33}$$

We see that we have to pay the price for the transition from the nonrelativistic eq. (31) to its relativistic descendant (29). Irrespectively of whether the initial scalar operator $H^{(KG)}$ itself is Hermitian, quasi-Hermitian or pseudo-Hermitian (in correspondence to the respective "metric" $\eta = I$, $\eta > 0$ or merely non-singular in eq. (32)), its new Feshbach-Villars representation $H^{(FV)}$ remains non-Hermitian.

One may reverse this argument as follows. Once we discover the merits of a regularization (e.g., of any centrifugal-like singularity in $H^{(KG)}$) by its (e.g., constant-shift) complexification, an extension of our previous results to the new relativistic context becomes almost straightforward. In spite of the fact that any detailed analysis in this direction would already lie beyond the scope of our present paper, we must emphasize that the auxiliary diagonalization of $H^{(FV)}$,

$$\left(\begin{array}{cc} 0 & H^{(KG)} \\ 1 & 0 \end{array} \right) \cdot \left(\begin{array}{c} |n_1^{(\pm)}\rangle \\ |n_2^{(\pm)}\rangle \end{array} \right) = E_n^{(\pm)} \left(\begin{array}{c} |n_1^{(\pm)}\rangle \\ |n_2^{(\pm)}\rangle \end{array} \right) \tag{34}$$

remains feasible and facilitated by its partitioning which means that

$$H^{(KG)}\,|n_2^{(\pm)}\rangle = \left[E_n^{(\pm)}\right]^2 |n_2^{(\pm)}\rangle, \qquad |n_1^{(\pm)}\rangle = E_n^{(\pm)}\,|n_2^{(\pm)}\rangle\,. \tag{35}$$

Hence, for all the positive operators $H^{(KG)} > 0$ and for all the present SUSY and/or PTSY regularization purposes we may work with the modified Schrödinger-type equation (35), knowing that the superscript $^{(\pm)}$ indicates merely the sign of the real "energy" $E_n^{(\pm)} \equiv \pm|E_n^{(\pm)}|$. In parallel, one still has to keep in mind that even for all the Hermitian $H^{(KG)}$ with $\eta = I$, the FV-time-evolution itself remains pseudo-unitary [34] since

$$\mathcal{P} = \left(\begin{array}{cc} 0 & I \\ I & 0 \end{array} \right) = \left[\frac{1}{\sqrt{2}} \left(\begin{array}{cc} I & I \\ I & -I \end{array} \right) \right] \left(\begin{array}{cc} I & 0 \\ 0 & -I \end{array} \right) \left[\frac{1}{\sqrt{2}} \left(\begin{array}{cc} I & I \\ I & -I \end{array} \right) \right] . \tag{36}$$

Thus, the quantized Klein-Gordon system may be used and presented as an archetypal physical PTSY model [35, 36], weakening the above limitation of our scope to the mere non-relativistic cases.

6 Concluding remarks and open questions

Let us summarize that without a transition to the language of PTSY QM, virtually all the central oscillators have a perceivably different mathematical interpretation at $D = 1$ (in one dimension) and at $D > 1$ (when there emerges a strongly singular centrifugal force in eq. (1)). After this transition, the mathematical difference disappears with the re-emergence of the safely normalizable bound states of an even quasi-parity. In the new language, it is less puzzling and purely technical to select and characterize the unphysical character of all the quasi-even states at all the higher spatial dimensions $D \neq 1$.

We reminded the readers that the difference between the present and vanishing centrifugal singularity (at $D > 1$ and $D = 1$, respectively) becomes even much more unpleasant after one starts working in the formalism of SUSY QM. We reviewed briefly a few standard attempts at a resolution of this problem (they did not prove too satisfactory, anyhow) and showed that the analytic continuation techniques mediated by the above-mentioned PTSY mathematical formalism offer one of the best regularization recipes. In particular, it enabled us to complete our older results and to characterize the PTSUSY structure of the spiked harmonic oscillator models *at any complex value* of the strength $\alpha^2 - 1/4$ of their centrifugal-like spike.

Of course, the study of this subject is far from being completed. We have outlined one of the eligible directions of the future research by extending the scope of the present paper to the exactly solvable angular equations. As we mentioned, their importance lies in their indispensable role in the exact solvability of several Hermitian models (viz., in the three-body Calogero model and in the two-dimentional Smorodinsky-Winternitz non-central oscillator etc). In such a setting we might emphasize that within the broader framework of PTSY QM these angular equations need not even remain exactly solvable at all.

A few further open question may be also formulated in the area of a less direct overlap between SUSY QM and PTSY QM. As we emphasized, the relativistic Klein-Gordon equation does not provide a fully consistent description of a spinless particle but still, it represents one of the phenomenologically most valuable benchmark models in PTSY QM. In the two forthcoming appended remarks, we intend to underline the existence of the new mathematical challenges concerning the viability of its possible transfer into the (possibly, regularized) formalism of SUSY QM.

6.1 Feshbach-Villars Hamiltonians and SUSY

As we already mentioned, there exist several close formal parallels between the Schrödinger and Klein-Gordon equations in the present regularization and analytic-continuation context. The extent of these parallels is less clear when we start speaking an algebraic language. For definiteness of a brief exposition of this problem, let us assume that in a way paralleling the constructions of the non-relativistic SUSY QM, the left (Hermitian or non-Hermitian and regular or singular) Klein-Gordon operator is factorizable, $H_{(L)}^{(KG)} = b \cdot a$. This means that we may also factorize the

two by two Feshbach-Villars matrix operator postulating, for example, that

$$H_{(L)}^{(FV)} = \left(\begin{array}{cc} 0 & H_{(L)}^{(KG)} \\ I & 0 \end{array} \right) = B \cdot A, \quad A = \left[\begin{array}{cc} d & 0 \\ 0 & a \end{array} \right], \quad B = \left[\begin{array}{cc} 0 & b \\ c & 0 \end{array} \right], \quad c \cdot d = I\,. \tag{37}$$

In a subtle way, such a postulate proves mathematically inconsistent. Indeed, in the usual SUSY framework, operators (37) have to be inserted in the above-mentioned general definition (3) of the supercharges $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$, respectively, giving

$$S_{SUSY,A} = \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \end{array} \right], \qquad S_{SUSY,B} = \left[\begin{array}{cccc} 0 & 0 & 0 & b \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \tag{38}$$

and obeying the same graded algebra as above. Routinely, the partner Feshbach-Villars Hamiltonian becomes defined by the formula

$$H_{(R)}^{(FV)} = \left(\begin{array}{cc} 0 & H_{(R)}^{(KG)} \\ I & 0 \end{array} \right) = A \cdot B, \quad H_{(R)}^{(KG)} = d \cdot b, \quad a \cdot c = I\,. \tag{39}$$

This means that we need to find the quadruplet of operators a, b, c and d where, in a suitable basis, the initial or "input" factors a and b may be visualized as having the respective annihilation- and creation-operator-type one-diagonal and infinite-dimensional matrix forms,

$$a = \left(\begin{array}{ccccc} 0 & a_0 & 0 & \ldots & \\ & 0 & a_1 & 0 & \ldots \\ & & & \ddots & \end{array} \right), \qquad b = \left(\begin{array}{ccccc} 0 & \ldots & & & \\ b_0 & 0 & \ldots & & \\ 0 & b_1 & 0 & \ldots & \\ & & \ddots & & \end{array} \right). \tag{40}$$

In this language the constraint $a \cdot c = I$ implies that in the matrix

$$c = \left(\begin{array}{ccccc} x_0 & x_1 & x_2 & x_3 & \ldots \\ 1/a_0 & 0 & \ldots & & \\ 0 & 1/a_1 & 0 & \ldots & \\ 0 & 0 & 1/a_2 & 0 & \ldots \\ & & & \ddots & \end{array} \right) \tag{41}$$

the first row remains arbitrary. Unfortunately, this freedom is not enough for the existence of a solution of our original requirement $c \cdot d = I$. Indeed, even if we omit the upper-left element of the latter matrix requirement, we get the unique solution for this weaker constraint,

$$d = \left(\begin{array}{ccccc} 0 & a_0 & 0 & \ldots & \\ & 0 & a_1 & 0 & \ldots \\ & & & \ddots & \end{array} \right) \equiv a\,. \tag{42}$$

Vice versa, we may evaluate the product

$$c\dot{d} = \begin{pmatrix} 0 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & \\ 0 & 0 & 1 & 0 & \dots \\ & & & \ddots & \end{pmatrix} \equiv \Pi = \Pi^2 \neq I\,. \tag{43}$$

This completes our constructive proof that the requirement $c{\cdot}d = I$ has no solution at all. In the other words, the only available Klein-Gordon-inspired super-Hamiltonian

$$H_{KGSUSY} = \left[\begin{array}{cccc} 0 & H^{(KG)}_{(L)} & 0 & 0 \\ \Pi & 0 & 0 & 0 \\ 0 & 0 & 0 & H^{(KG)}_{(R)} \\ 0 & 0 & I & 0 \end{array} \right], \qquad H^{(KG)}_{(L,R)} = -\frac{d^2}{dx^2} + V^{(KG)}_{(L,R)}(x)\,. \tag{44}$$

connects merely the right Klein-Gordon field with an artificial, "constrained" left SUSY partner which might only be interpreted as a quasi-Klein-Gordon system equipped with an additional projection-operator constraint $n_1^{(\pm)} = \Pi\, n_1^{(\pm)}$ at all n. Its physical interpretation remains unclear.

6.2 Generalized time-evolution of $k-$th order

When we compare the above Schrödinger and Klein-Gordon SUSY QM constructions, we may notice that they may be interpreted, quite formally, as the first two elements of an infinite hierarchy of the $k-$th order time-evolution equations

$$(i\,\partial_t)^k\, \Psi^{(k)}(x,t) = H^{(k)}\, \Psi^{(k)}(x,t), \qquad k = 1, 2, 3, \dots\,. \tag{45}$$

In a way which extends and parallels our previous notation we may put

$$\varphi_j = i\,\partial_t\, \varphi_{j+1}, \quad j = 1, 2, \dots, k-1, \quad \varphi_k = \Psi^{(k)}(x,t) \tag{46}$$

and arrange eq. (45) in the $k-$dimensional matrix form

$$i\,\partial_t \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_k \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & H^{(k)} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_k \end{pmatrix}. \tag{47}$$

This may be read as the new matrix problem $i\,\partial_t \vec{\varphi}^{(k)} = \mathcal{H}^{(k)}\, \vec{\varphi}^{(k)}$ where the matrix of the Hamiltonian obeys the pseudo-Hermiticity condition

$$\left[\mathcal{H}^{(k)}\right]^\dagger = \mathcal{P}\, \mathcal{H}^{(k)}\, \mathcal{P}^{-1}, \qquad \mathcal{P} = \begin{pmatrix} 0 & 0 & \dots & 0 & \eta \\ 0 & \dots & 0 & \eta & 0 \\ \vdots & & \iddots & & \vdots \\ 0 & \eta & 0 & \dots & 0 \\ \eta & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{48}$$

The metric operators η must be Hermitian and invertible and may become positive or equal to unit operators in the respective quasi-Hermitian or Hermitian special cases. In this sense the Schrödinger ($k = 1$) and Klein-Gordon ($k = 2$) evolution equations may be viewed as special cases of a more general scheme without, of course, any immediate applications within quantum physics at $k \geq 3$.

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